Let us estimate the magnitude of the pressure in a steel vessel for $R/r = 100$.

The quantity $p - \sigma = 2\tau$, where τ is the shear strength. For steel we take $\tau = 500$ MPa, $\alpha = 12 \cdot 10^{-6}$ 1/K, $E = 2 \cdot 10^5$ MPa.

For $R/r = 100$, we obtain $p(100) = 9200$ MPa at $T_0 \approx 200$°K from (5).

Taking account of other effects (the final volume of lead in the interlayers, its expansion during melting, the hardening of steel under pressure, etc.) radically complicates the problem but does not eliminate the divergence in the pressure during self-filling of the vessel. Taking account of the compressibility of vessel layer material (steel) diminishes the magnitude of the pressure for self-filling the vessel, the question of the pressure discrepancy at the center of the vessel remains open here.

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ELASTIC–PLASTIC BEHAVIOR OF A MATERIAL, TAKING MICROINHOMOGENEITIES INTO ACCOUNT

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The construction of a theory of plasticity satisfactorily describing the singularities in polycrystalline material behavior is one of the urgent problems of the mechanics of a deformable solid. The regularities in the plastic deformation of a polycrystalline aggregate are statistical in nature since they are the result of interaction of a large number of randomly oriented crystals. This paper is a further development of the results obtained in [1-3].

As a rule, the influence of a "physical" microinhomogeneity (elastic and plastic anisotropies of the crystallites) has been examined in investigations of a similar kind [4, 5]. In the present paper the influence of both the "physical" and structural inhomogeneities of a polycrystalline body (associated with the spread in the dimensions of its crystallite components) on the nature of the elastic–plastic deformation is analyzed.

1. As is known, all real metals are polycrystalline media, i.e., are conglomerates of randomly oriented subcrystals (grains) that are characterized by a definite spatial arrangement of the crystalline lattice.

The physical model of a polycrystal described in [1, 2] is taken for the subsequent investigations. The macroscopic stress–strain state of the medium is assumed homogeneous, the mechanism of plastic deformation is considered to be translational slip in the crystallites forming the aggregate.

It is known [5, 6] that metals can experience considerable deformation without the formation of cracks so that crystals in mutual contact in an undeformed material retain this contact in the whole deformation stage. This means that the equilibrium equations and the strain compatibility conditions are satisfied at each point of the polycrystalline material:

$$\nabla_j \sigma_{ij} = 0; \tag{1.1}$$

$$\mathscr{E}_{ijlkmn} \nabla_m \nabla_n \varepsilon_{lm} = 0. \tag{1.2}$$

Here $\mathscr{E}_{ijlkmn} = \delta_{ilm}\delta_{jkn}$, where δ_{ilm} is the unit Levi–Civita pseudotensor, and $\nabla_j = \partial/\partial x_j$.

The relationships

$$\sigma_{ij} = \langle \sigma_{ij} \rangle, \tag{1.3}$$

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are satisfied on the polycrystalline surface, and the stress and strain in the grains are related by Hooke's law

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + e_{ij}, \tag{1.4}$$

where e_{ij} is the plastic deformation fields, and S_{ijkl} is the grain elastic pliability tensor dependent on the crystallite orientation and the kind of crystal lattice of the material.

Let us determine the relation between the stress and incompatible strain fields in the polycrystal by using the solution of the fundamental problem of elasticity theory (1.1)-(1.3) in stresses. Substituting (1.4) into (1.2), we obtain

$$B_{ijrs}(\nabla)(\sigma_{rs} - \langle\sigma_{rs}\rangle) = -f_{ij}; \tag{1.5}$$

$$B_{ijrs}(\nabla) = \mathscr{E}_{ijlkmn}\ C_{lkrs}\nabla_m\nabla_n; \tag{1.6}$$

$$f_{ij} = \mathscr{E}_{ijlkmn}\nabla_m\nabla_n\varkappa_{lk}; \tag{1.7}$$

$$\varkappa_{lk} = D_{lkrs}\sigma_{rs} + e_{lk} - \langle\varepsilon_{lk}\rangle + C_{lkrs}\langle\sigma_{rs}\rangle, \tag{1.8}$$

where C_{ijlk} is the material elastic pliability tensor assumed known, $D_{lkrs} = S_{lkrs} - C_{lkrs}$ is the component of the tensor S_{lkrs} that depends on grain orientation, and $\varkappa_{lk}$ is the incompatible strain field.

It has been shown in [7] that the field f_{ij} cannot be arbitrary and should satisfy the Krener gauge condition

$$\nabla_j f_{ij} = 0. \tag{1.9}$$

It can be shown by direct substitution that this condition is satisfied in this case.

Determining the Green's tensor $R_{lkrs}(\mathbf{r})$ for the differential operator $B_{ijlk}(\nabla)$ from the equation

$$B_{ijlk}(\nabla)R_{lkrs}(\mathbf{r}) = (1/2)(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})\delta(\mathbf{r}),$$
$$R_{ijrs}(\mathbf{r}) \to 0 \quad \text{for } r \to \infty,$$

we find the relation between the stress in the polycrystalline grains and the incompatible strain field from (1.5)

$$\sigma_{ij}(\mathbf{r}) - \langle\sigma_{ij}\rangle = \int_V \mathscr{E}_{pqlkmn}\nabla_m R_{ijpq}(\mathbf{r}-\mathbf{r}')\nabla_m\varkappa_{lk}(\mathbf{r}')\,dV'. \tag{1.10}$$

The integral

$$J_{ij} = \int_{S\infty} \mathscr{E}_{pqlkmn}R_{ijpq}(\mathbf{r}-\mathbf{r}')\nabla_n\varkappa_{lk}(\mathbf{r}')\,n_m dS,$$

which equals zero has been discarded in (1.10), and the fact is taken into account that the function $\nabla_m R_{ijpq}(\mathbf{r}-\mathbf{r}')$ is regular in the domain V, where V is the specimen volume.

For a macroisotropic medium the operator $B_{ijlk}(\nabla)$ has the form

$$B_{ijlk}(\nabla) = (p_0 + 2s_0)\delta_{lk}(\nabla^2\delta_{ij} - \nabla_i\nabla_j) - 2s_0\nabla^2\delta_{il}\delta_{jk},$$

and the Green's tensor R_{ijpq} is defined by the relationship

$$R_{ijpq}(\rho) = \frac{\mu_0}{16\pi}[I_{ijpq}\nabla^2\rho - \zeta\delta_{pq}(\delta_{ij}\nabla^2\rho - \nabla_i\nabla_j\rho)]. \tag{1.11}$$

Here

$$I_{ijpq} = \frac{1}{2}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}); \quad s_0 = \frac{1}{4\mu_0}; \quad \zeta = \frac{1}{4\nu_0} = 10g_0; \quad \rho = |\mathbf{r}-\mathbf{r}'|;$$

λ_0, μ_0, ν_0 are the Lamé parameters and Poisson ratio of a continuous medium with an elastic pliability field C_{ijkl}.

For R_{ijpq} defined by (1.11), the relationship (1.10) satisfies the equilibrium equations (1.1) and the boundary conditions (1.3), and therefore, is a solution of the funamental problem of elasticity theory for a polycrystal.

The relationship (1.10) is exact in the sense that it has been obtained without any assumptions about the form of the incompatibility field and is valid for all the polycrystal grains.

2. Let us assume that all the polycrystal grains are simply connected, contain centers of gravity inside, and the crystallite shape is independent of the orientation of the crystallographic axes. Then a passage from the physical model of a polycrystal with a spatial crystallite arrangement to its statistical model with average grains being distinguished only by orientations and size [1, 2] is possible. The incompatibility fields averaged with respect to Ω_l (i.e., the set of grains, orientations, and dimensions located in an infinitesimal neighborhood of a point corresponding to the orientation and dimension of a certain fixed gain in the space of orientations and dimensions) turn out to be isotropic in the statistical model of the medium. Hence, under the assumption of isotropy of the macroscopic elastic properties of the polycrystal, the integral in the right side of (1.10) can be evaluated, and the equation mentioned becomes

$$\sigma_{ij} - \langle\sigma_{ij}\rangle = L_{ijkl}\varkappa_{kl}(\omega, l), \tag{2.1}$$

where

$$L_{ijkl} = -\frac{2\mu_0}{3}[(1+4g_0)I_{ijkl} - (1-12g_0)J_{ijkl}],$$
$$\varkappa_{kl}(\omega, l) = \varkappa_{kl}(0|\omega, l),\ J_{ijkl} = \delta_{ij}\delta_{kl}, \tag{2.2}$$

$\omega(\varphi, \psi, \theta)$ is the grain orientation in the space of Euler angles (φ, ψ, θ), and l is the characteristic dimension of the grain.

From the relations (2.1) and (1.8) we obtain an equation relating the stress and plastic deformation fields in the average grain to the macroscopic stress–strain state of the polycrystal:

$$Q_{ijkl}\sigma_{kl} - L_{ijkl}e_{kl} = \xi_{ij}; \tag{2.3}$$

$$Q_{ijkl} = I_{ijkl} - L_{ijrs}D_{rskl}; \tag{2.4}$$

$$\xi_{ij} = (I_{ijkl} + L_{ijrs}C_{rskl})\langle\sigma_{kl}\rangle - L_{ijkl}\langle e_{kl}\rangle, \tag{2.5}$$

where ξ_{ij} is the parametric tensor of the macroscopic stress–strain state.

We turn to the strengthening law [2]

$$\tau_0^\alpha = \tau_0(1 + a_{\alpha\beta}w_\beta) \tag{2.6}$$

to obtain the plastic shear equation in the effective polycrystal grain, where τ_0 is the critical tangential stress in the undeformed crystal, τ_0^α is the decisive tangential stress in the slip system α; $w_\alpha = \int |d\lambda|$ is the integrated shear in the system α; $a_{\alpha\beta}$ is the element of the strengthening matrix. Experiments [6] show that the critical tangential stress τ_0 depends on the magnitude of the grain. The form of this dependence can be determined by using the known Hall–Patch equation for the macroscopic elastic limit of the polycrystal

$$\tau_* = \tau_0^* + k_* l^{-n},$$

where l is the grain dimension, τ_0^* is the friction stress equal to the elastic limit of a polycrystal consisting of a single monocrystal, and k_* is a constant of the material that is associated with strain propagation through the grain boundary. The exponent n for BCC metals is usually 0.5, and for FCC and HCP metals can take the values 0.5 and 1.0.

The elastic limit in shear for a polycrystalline material does not agree with the elastic limit of a grain taken separately because of the elastic anisotropy of the components of the crystallite aggregates. The ratio of the macroscopic and local elastic limits $\eta = \tau_*/\tau_0$ depends on the kind of crystalline lattice and the elastic constants of the crystal [1, 8].

The reasoning presented permits determination of the dependence of τ_0 on the grain dimension

$$\tau_0(l) = \frac{1}{\eta}(\tau_0^* + k_* l^{-n}). \tag{2.7}$$

Going over from finite quantities to their increments in (2.6) and the substituing of σ_{ij} from (2.3) with the rule for transformation of tensors during passage from crystallographic axes to axes coupled to the slip systems of the crystal [2] taken into account, we obtain the equation of plastic shears in an effective grain with dimension l and orientation ω:

$$m_{\alpha\beta}\dot{\lambda}_{\beta} = w_{ij}^{\alpha}\dot{\xi}_{ij}. \quad (2.8)$$

Here

$$m_{\alpha\beta} = t_{ij}^{\alpha}W_{ijkl}t_{kl}^{\beta} + \tau_0(l)\,a_{(\alpha)(\beta)}\,\text{sign}\,\tau_{(\alpha)}\,\text{sign}\,\tau_{(\beta)};$$
$$W_{ijkl} = 2(L_{ijrs}D_{rspq} - I_{ijpq})^{-1}L_{pqlk};\quad w_{ij}^{\alpha} = t_{kl}^{\alpha}(I_{klij} - L_{klpq}D_{pqij})^{-1};$$

τ_{ij}^{α} is the matrix of coordinate transformation from the crystallographic axes to axes coupled to the slip system with number α in the effective grain, and $\dot{\lambda}_{\alpha}$ is the plastic shear in the system. There is no summation over the subscripts in parentheses. The operator $(\cdot)$ means the passage from finite quantities to the infinitesimal increments.

When examining unstrengthened material ($a_{\alpha\beta} = 0$) the impression can occur that the plastic shears determined from the system (2.8) are independent of the dimension of the effective grain. However, this is not actually so since the mentioned system of equations should be solved under the condition $\tau_{\alpha} = \tau_0^{\alpha}(l)$.

Determining the plastic shears from (2.8), we find the dependence between $\dot{\sigma}_{ij}$ and the parametric tensor of the macroscopic stress strain state

$$\dot{\sigma}_{ij}(\omega, l) = M_{ijrs}(\omega, l)\dot{\xi}_{rs}, \quad (2.9)$$

where

$$M_{ijrs}(\omega, l) = Q_{ijmn}^{-1}\left[2L_{mnpq}t_{pq}^{\alpha}m_{\alpha\beta}^{-1}(\omega, l)\,t_{kl}^{\beta}Q_{klrs}^{-1} + I_{mnrs}\right] \quad (2.10)$$

from (1.4) and (2.3) with the known relationship between the plastic shears and deformations taken into account.

Summation in (2.10) is over those values of α and β that correspond to active slip systems [2].

For a microhomogeneous material ($D_{ijpq} = 0$) under the condition that τ_0 is independent of the grain dimension, (2.9) agrees with the known Krener–Budiansky–Wu equation [5].

The relation between the macroscopic stresses and strains in a polycrystalline material (the plasticity law) is determined by averaging (2.9) over the domain of grain dimension distribution and the set of orientations with given probability density $p(\omega, l)$.

Taking account of (2.5), we obtain

$$\langle\dot{\sigma}_{ij}\rangle = \chi_{ijrs}\langle\dot{\varepsilon}_{rs}\rangle,$$
$$\chi_{ijrs} = \langle M_{ijkl}(\omega, l)\rangle_{\omega,l}(I_{klmn} - L_{klpq}C_{pqmn})^{-1}\langle M_{mnuv}\rangle L_{uvrs},$$

where χ_{ijrs} is the tensor of instantaneous plastic moduli of a polycrystal at this stage of the history of strain. The form of the tensor χ_{ijrs} depends on the trajectory of a point in strain (or stress) space.

3. Let us consider a computation of the strain curves for aluminum of 99.99% purity. Let the grain distribution with respect to the orientations and dimensions in the aggregate be given by the probability density

$$p(\omega, l) = p_1(\omega)p_2(l),\ p_1(\omega) = 1/8\pi^2 \quad (3.1)$$
$$(0 \leqslant \varphi \leqslant \pi,\ 0 \leqslant \psi \leqslant 2\pi,\ 0 \leqslant \theta \leqslant 2\pi),$$
$$p_2(l) = \frac{c}{\sqrt{2\pi d}}e^{-\frac{(l-\langle l\rangle)^2}{2d}} \qquad (l_1 \leqslant l \leqslant l_2),$$

where l_1 = 0.1415 mm; l_2 = 0.353 mm; $\langle l\rangle = 0.2473$ mm; and c is a normalizing constant dependent on d.

To determine the constants τ_0^* and k_* in (2.8), the curve 1 (Fig. 1) corresponding to the experimental dependence of the elastic limit of aluminum on the mean grain dimension [6] is used. Curves 2 and 3 correspond to the approximation of experimental data to the Hall–Patch

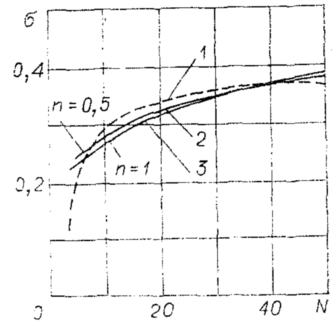

Fig. 1

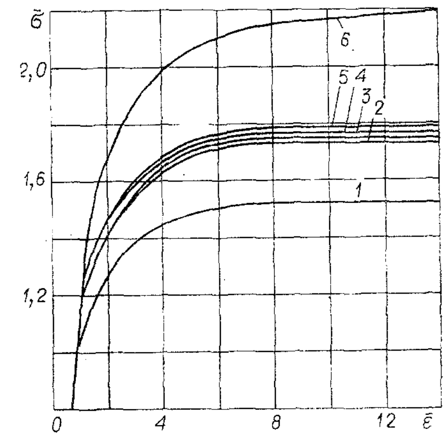

Fig. 2

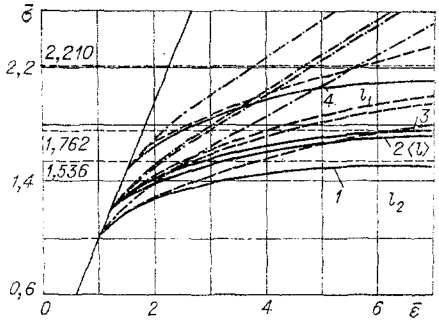

Fig. 3

dependence for the exponents n = 0.5 and 1.0, respectively. It follows from the graph that the Hall–Patch curve with exponent n = 0.5 approximates the experimental curve better. This also agrees with the experimental data [6]. Taking the above into account, we found that $\tau_0^* = 0.066$ kg/mm^2, $k_* = 0.122$ kg/mm$^{3/2}$.

In order to simplify the calculations, the strain curves were analyzed for the simple tension case. The results of the computations for the model proposed for the polycrystalline material with ideally plastic grains are presented in Fig. 2 in the coordinates $\bar{\varepsilon} = \varepsilon/\varepsilon_0$, $\bar{\sigma} = \sigma/\sigma_0$, where σ_0 and ε_0 are the elastic limit and the corresponding elastic strain of a polycrystal consisting of grains of dimension l_2.

Curve 1 characterizes the behavior of crystallites whose dimension corresponds to the right end of the distribution interval (coarse grains), and 6 to the left end (fine grains). The dependence 2 characterizes the behavior of grains of the most probable dimension $\langle l \rangle$ Curves 3-5 are strain patterns of a polycrystalline aggregate in which the crystallite dimension distribution is described by the probability density (3.1) with different values of the rms deviation $\sqrt{d} = 0.025$, $\sqrt{d} = 0.05$, $\sqrt{d} = 0.1$, respectively.

The influence of isotropic strengthening (according to Taylor) on the strain diagram of a polycrystal is exhibited in Fig. 3, where curves 1-4 correspond to curves 1, 2, 5, 6 in Fig. 2. The dashed lines correspond to the value of the quantity b/G = 0.02 (b is the isotropic strengthening factor, and G is the shear modulus), and the dash-dot line to b/G = 0.10.

It follows from the dependences presented that the mean grain dimension exerts the fundamental influence on the behavior of a polycrystalline aggregate consisting of crystallites of different dimensions. The dependence of the strain diagram on the nature of the grain dimension distribution turns out to be negligible.

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PARAMETRIC RESONANCE IN A STRATIFIED FLUID

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Parametric resonance is one of the widespread types of instability of mechanical systems. A somewhat broader class of phenomena is called parametrically excited oscillations. The mathematical definition of this class of oscillations is ordinarily given [1] for systems whose equations of motion reduce to ordinary differential equations in the time. Parametric oscillations are related to the periodic dependence of the coefficients (parameters) of these equations on the time. Such oscillations are distinct from forced oscillations for which the explicit time dependence is contained only additively, in the form of periodic forces, in the equations. The Mathieu equation and its generalization are a standard example of parametric oscillation equations. The experimental work of Faraday [2], in which the oscillations of a free fluid surface in a vessel were studied, was the first investigation of parametric oscillations. However, mainly applications to solid and elastic bodies [1, 3, 4] were developed later. The exception is the problem of the oscillations of a free fluid surface in a vertically oscillating vessel. It has been shown [5-7] that in a linear approximation, the displacement of a free surface reduces to a Mathieu equation, and resonance frequencies therefore exist for which the surface turns out to be unstable. Taking account of the viscosity in this problem is presented in [8]. Only in the past decade have investigations been started on the parametric instabilities of more complicated flows. Parametric resonance in convection problems was studied in [9, 10]. The stability of Rossby waves was investigated in [11-14]. The papers [15, 16] are devoted to the instability of internal waves in a stratified fluid. A number of considerations on the possibility of the growth of fine-scale perturbations in the internal wave background is presented in [15].

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